

ALMOST SUBMANIFOLD STRUCTURES

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1. Introduction

The purpose of this paper is the investigation of certain structures on an n -dimensional C^∞ manifold determined by a second order connection, which we call almost submanifold structures. The case of an almost submanifold structure satisfying a certain additional condition is called an almost hypersurface structure, and is studied in detail. An almost hypersurface structure on a manifold allows us to treat the manifold almost as if it were a hypersurface of a second manifold. For example we may discuss the mean curvature and directions of curvature on a manifold bearing an almost hypersurface structure. We show that an almost hypersurface structure is integrable (isometrically imbeddable in a Euclidean space) if and only if the curvature tensor of the structure vanishes.

Various conditions are obtained that there exists a submanifold whose geodesics are also second order geodesics of a second order connection, and the mean curvature vector of and almost submanifold structure is investigated.

2. Preliminary remarks

In this section we will outline the results of [1] and [2] utilized in the main body of this paper. The notation utilized is essentially that of [1] and [3] with the summation convention employed on lower case Latin indices.

If 2M denotes the third term of the extended sequence [2] of a manifold $M (\equiv {}^0M)$:

$$(1) \quad {}^0M \xleftarrow{{}^1_0\pi} {}^1M \xleftarrow{{}^2_1\pi} {}^2M \xleftarrow{{}^3_2\pi} \dots,$$

then a second order connection [1] on M is a connection on the bundle ${}^2_0\pi: {}^2M \rightarrow M$, which naturally induces a connection on \bar{M} (which we called the first order connection induced on M). If ${}^1_0\pi_*$ denotes the tangent map of ${}^1_0\pi: {}^1M \rightarrow M$ (${}^1M \equiv TM$), and K is the connection map of the induced first order connection, then TTM and consequently 2M may be given a vector bundle structure over M relative to these maps. If HM and VM denote the horizontal and vertical subbundles of 2M respectively, then

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$$(2) \quad \overset{1}{\pi}_* : HM_p \rightarrow TM_{\overset{1}{\pi}(p)}, \quad K : VM_p \rightarrow TM_{\overset{1}{\pi}(p)}$$

are isomorphisms of each $p \in TM$; and if (U, ϕ) is a coordinate chart of M , then there are naturally determined bases $\{X_i^h\}$ and $\{X_i^v\}$ of the horizontal and vertical subbundles respectively, and coordinates of 2M (called vector bundle coordinates) relative to (U, ϕ) and the first order connection on M .

A second order connection on M determines a covariant differentiation of a section A of $\overset{2}{\pi} : {}^2M \rightarrow M$ with respect to a vector field X on M (here and throughout the remainder of the paper we identify the horizontal subbundle of 2M with TM) which has the local form

$$(3) \quad \tilde{\nabla}_X A = \xi^j \left(\frac{\partial A^{0i}}{\partial x^{0j}} + \Gamma_{jk}^{0i} A^{0k} \right) X_i^h + \xi^j \left(\frac{\partial A^{1i}}{\partial x^{0j}} + \Gamma_{jk}^{1i} A^{0k} + \Gamma_{jk}^{1i} A^{1k} \right) X_i^v,$$

where $X = \xi^j \partial / \partial x^j$, and $A = A^{0i} X_i^h + A^{1i} X_i^v$.

If X and Y are C^∞ vector fields on M , and ξ is a vertical vector field on M , i.e., a C^∞ map $\xi : M \rightarrow VM$, then decomposing $\tilde{\nabla}$ into horizontal and vertical components yields

$$(4) \quad \tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \quad \tilde{\nabla}_X \xi = D_X \xi,$$

where the horizontal component $\nabla_X Y$ is the covariant derivative of the induced first order connection, the vertical component $\alpha(X, Y)$, which we call the second fundamental form of $\tilde{\nabla}$, is bilinear, and the vertical component $D_X \xi$ is a connection in the vertical bundle.

If γ is a C^∞ curve of M , then γ' will denote the canonical lift of γ to TM , and γ'' the canonical lift of γ' to ${}^2M \subset TTM$.

3. Almost submanifold structures

Suppose that g is a fiber metric on $\overset{2}{\pi} : {}^2M \rightarrow M$, and that $\mathfrak{X}(M)$ and $\mathfrak{X}^v(M)$ denote the modules of C^∞ horizontal and vertical vector fields on M respectively. If $\tilde{\nabla}$ is a second order connection on M , then for all $X, Y \in \mathfrak{X}(M)$ and $\xi \in \mathfrak{X}^v(M)$ there is a unique field in $\mathfrak{X}(M)$, which we denote by $\mathcal{A}_\xi(X)$ such that

$$(5) \quad g(\mathcal{A}_\xi(X), Y) = g(\alpha(X, Y), \xi).$$

Using \mathcal{A} we define an operator ∇' such that

$$(6) \quad \nabla'_X Y = \nabla_X Y + \alpha(X, Y), \quad \nabla'_X \xi = -\mathcal{A}_\xi(X) + D_X \xi,$$

where ∇, α and D are as in (4). Such an operator satisfying the additional condition (5) will be called an almost submanifold structure or AS -structure on M . Thus by construction we see that to each pair consisting of a fiber metric

on ${}^2\pi: {}^2M \rightarrow M$ and a second order connection on M there corresponds a unique AS-structure on M .

Suppose that for a given fiber metric g on ${}^2\pi: {}^2M \rightarrow M$ and second order connection $\tilde{\nabla}$, the first order connection induced on M by $\tilde{\nabla}$ is metric with respect to the metric on M obtained by restricting g to the horizontal subbundle of 2M , and that the connection on the vertical bundle induced by $\tilde{\nabla}$ is metric with respect to the metric obtained by restricting g to the vertical subbundle of 2M . If in addition the torsion $\widetilde{\text{Tor}}(X, Y)$ of $\tilde{\nabla}$ with respect to any X, Y , given by

$$(7) \quad \widetilde{\text{Tor}}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y],$$

vanishes, we will say that $\tilde{\nabla}$ is Riemannian with respect to g .

Theorem 1. *If $\tilde{\nabla}$ is Riemannian with respect to a fiber metric g , with the additional property that vertical and horizontal vector are orthogonal at each point of M , then ∇' is Riemannian with respect to g .*

Proof. Letting $A = A^h + A^v$ where A^h and A^v are the horizontal and vertical components of A we see that since horizontal and vertical vectors are orthogonal and $\tilde{\nabla}$ Riemannian,

$$\begin{aligned} Xg(A, B) &= Xg(A^h, B^h) + Xg(A^v, B^v) \\ &= g(\nabla_X A^h, B^h) + g(A^h, \nabla_X B^h) + g(D_X A^v, B^v) + g(A^v, D_X B^v). \end{aligned}$$

From (5) it follows that

$$\begin{aligned} g(\alpha(X, A^h), B^v) + g(\mathcal{A}_{B^v}(X), A^h) &= 0, \\ g(\alpha(X, B^h), A^v) + g(\mathcal{A}_{A^v}(X), B^h) &= 0, \end{aligned}$$

so that

$$Xg(A, B) = g(\nabla'_X A, B) + g(A, \nabla'_X B).$$

Since

$$\text{Tor}'(X, Y) = \nabla'_X Y - \nabla'_Y X - [XY] = \widetilde{\text{Tor}}(X, Y) = 0,$$

we see that ∇' is Riemannian with respect to g .

Theorem 2. *If the AS-structure ∇' is Riemannian with respect to the fiber metric g , then the first order connection induced by ∇' is Riemannian with respect to the metric induced by g , and $\alpha(X, Y) = \alpha(Y, X)$.*

Proof. Since ∇' is Riemannian, using (6) we have

$$\nabla'_X Y - \nabla'_Y X - [X, Y] = \nabla_X Y - \nabla_Y X + \alpha(X, Y) - \alpha(Y, X) = 0,$$

hence

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0, \quad \alpha(X, Y) = \alpha(Y, X).$$

We define the first vertical space $V_1(x)$ of an AS -structure at $x \in M$ by

$$(8) \quad V_1(x) = \text{span} \{ \alpha(X, Y) \mid X, Y \in M_x \}.$$

If $V_1(x)$ has maximum dimension l at any point $x \in M$, we call l the pseudocodimension of M .

4. Almost hypersurface structures

We first consider the case where the pseudocodimension of the AS -structure is 1. Let

$$(9) \quad \xi_x = \begin{cases} \frac{\alpha_x(X, Y)}{\|\alpha_x(X, Y)\|} & \text{if } \alpha_x(X, Y) \neq 0 \text{ for some } X, Y, \\ 0 & \text{if } \alpha_x \equiv 0. \end{cases}$$

If $h(X, Y) = \|\alpha(X, Y)\|$, then $\alpha(X, Y) = h(X, Y)\xi$. If we take $\mathcal{A}(X) = \mathcal{A}_\xi(X)$ for the ξ defined in (9), the AS -structure becomes

$$(10) \quad \nabla'_X Y = \nabla_X Y + \alpha(X, Y), \quad \nabla_X \xi = -\mathcal{A}(X) + D_X \xi,$$

where $(D_X \xi)_x = 0$ if $\xi_x = 0$, and we restrict ourselves to the case where \mathcal{A} is C^∞ henceforth.

On the basis of (10) we may define various notions analogous to those of a hypersurface. At a point $x \in M$ the mean curvature $H(x)$ is the trace of \mathcal{A}_x , and the total curvature $K(x)$ is the determinant of \mathcal{A}_x . If $\lambda_1, \dots, \lambda_n$ are the eigenvalues of \mathcal{A}_x , they are the principal curvatures at x , and the corresponding eigenvectors are the directions of curvature at x . If two vectors at x have the property that $g(\mathcal{A}(X), Y) = 0$, then they are conjugate; and if $g(\mathcal{A}(X), X) = 0$, then X is asymptotic. If $\mathcal{A} = \lambda \text{Id}$, then x is umbilical, etc.

We define the curvature tensor of the AS -structure ∇' in the usual manner as follows:

$$(11) \quad R'(X, Y)A = \nabla'_X \nabla'_Y A - \nabla'_Y \nabla'_X A - \nabla'_{[X, Y]} A,$$

and note that a standard calculation shows that the horizontal component of $R'(X, Y)Z$ is equal to

$$(12) \quad R(X, Y)Z + h(X, Z)\mathcal{A}(Y) - h(Y, Z)\mathcal{A}(X),$$

where $X, Y, Z \in \mathfrak{X}(M)$, and R is the curvature tensor of the induced first order connection. The vertical component is equal to

$$(13) \quad (\nabla_X h)(Y, Z)\xi - (\nabla_Y h)(X, Z)\xi + h(Y, Z)D_X \xi - h(X, Z)D_Y \xi.$$

Suppose that $R' \equiv 0$. Then from the horizontal component we have

$$(14) \quad R(X, Y)Z = h(Y, Z)\mathcal{A}(X) - h(X, Z)\mathcal{A}(Y)$$

or equivalently

$$(15) \quad R(X, Y)Z = g(\mathcal{A}(Y), Z)\mathcal{A}(X) - g(\mathcal{A}(X), Z)\mathcal{A}(Y) .$$

From the vertical component of $R'(X, Y)Z$ it follows that at those $x \in M$ where ξ is not continuous, $D\xi = 0$ (by definition) whence

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) .$$

At those $x \in M$ where ξ is C^∞ , we have $g(\xi, \xi)$ constant and thus

$$Xg(\xi, \xi) = 2g(D_X \xi, \xi) = 0 .$$

Hence $D_X \xi$ and ξ are orthogonal, which together with (13) written in the form

$$(\nabla_X h)(Y, Z)\xi - \nabla_Y h(X, Z)\xi = h(X, Z)D_X \xi - h(Y, Z)D_X \xi$$

implies that in either case

$$(16) \quad (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) ,$$

or alternately

$$(17) \quad (\nabla_X \mathcal{A})(Y) = (\nabla_Y \mathcal{A})(X) .$$

Hence in the case of an AS -structure of pseudocodimension 1, (6) becomes

$$(18) \quad \nabla'_X Y = \nabla_X Y + \alpha(X, Y) , \quad \nabla'_X \xi = -\mathcal{A}(X) .$$

Theorem 3. *On a manifold bearing a Riemannian AS -structure of pseudocodimension 1 such that for $X, Y, Z \in \mathfrak{X}(M)$ the horizontal component of $R'(X, Y)Z$ vanishes, the Ricci tensor is given by*

$$\text{Ri}(X, Y) = g(\mathcal{A}(X), Y) \text{tr } \mathcal{A} - g(\mathcal{A}^2(X), Y) .$$

Proof. By definition

$$\text{Ri}(XY) = \text{trace of the map } Z \rightarrow R(Z, X)Y$$

using (15) in a standard fashion we obtain the desired formula.

We will say that an AS -structure of pseudocodimension 1 on M is integrable if there is an isometric imbedding of M into R^{n+1} with second fundamental

tensor \mathcal{A} the tensor determined by the symmetric transformation of the AS -structure in (18).

Theorem 4. *A Riemannian AS -structure of pseudocodimension 1 on a connected simply connected manifold is integrable if and only if $R' \equiv 0$.*

Proof. Suppose that $R' \equiv 0$. Then from (15) and (17) we see that the symmetric linear transformation \mathcal{A} satisfies the Gauss-Codazzi equations. The integrability of the AS -structure on M then follows from the fundamental theorem for hypersurfaces [3]. On the other hand suppose that the AS -structure on M is integrable. Then \mathcal{A} satisfies the Gauss-Codazzi equations, and equations (15) and (17) imply that $R'(X, Y)Z = 0$ for arbitrary $X, Y, Z \in \mathfrak{X}(M)$. From (18) it follows that

$$\begin{aligned}
 (19) \quad R'(X, Y)\xi &= \nabla'_X \nabla'_Y \xi - \nabla'_Y \nabla'_X \xi - \nabla'_{[X, Y]}\xi \\
 &= (\nabla'_Y \mathcal{A})(X) - (\nabla'_X \mathcal{A})(Y) - \mathcal{A}(\text{Tor}(X, Y)) \\
 &\quad - \alpha(X, \mathcal{A}(Y)) + \alpha(Y, \mathcal{A}(X)) .
 \end{aligned}$$

Since the Codazzi equation (17) is satisfied, $(\nabla'_Y \mathcal{A})(X) - (\nabla'_X \mathcal{A})(Y) = 0$; and since ∇' is Riemannian, $\mathcal{A}(\text{Tor}(X, Y)) = 0$. Noting that $g(\alpha(X, Y), \xi) = g(\mathcal{A}(X), Y)$ we see that

$$\begin{aligned}
 g(\alpha(X, \mathcal{A}(Y)), \xi) &= g(\mathcal{A}(X), \mathcal{A}(Y)) \\
 &= g(\mathcal{A}(Y), \mathcal{A}(X)) = g(\alpha(Y, \mathcal{A}(X)), \xi) ,
 \end{aligned}$$

so that $R'(X, Y)\xi = 0$.

Remark. Although an imbedding of M , to within an isometry of R^{n+1} , is determined by an integrable AS -structure, an imbedded submanifold of R^{n+1} may be determined, within an isometry of R^{n+1} , by several AS -structures. Suppose that M admits a global nonvanishing C^∞ vector field and is imbedded in R^{n+1} with second fundamental form h . If ξ and ξ' are C^∞ unit vector fields on M (these exist globally since M admits a nonvanishing C^∞ vector field and a connection map K) take $\alpha(X, Y) = h(X, Y)\xi$ and $\alpha'(X, Y) = h(X, Y)\xi'$, then the AS -structures so obtained yield the same imbedding of M into R^{n+1} , to within an isometry of R^{n+1} .

Theorem 5. *Suppose that M bears a Riemannian AS -structure of pseudocodimension 1 having the properties that the vertical component of R' vanishes, and that the type number $t(x)$ of \mathcal{A} at each point $x \in M$ is constantly l on an open neighborhood U of M . Then through each $x \in U$ there passes a maximal submanifold S of dimension $n - l$ having the property that each geodesic of S is also a second order geodesic of S (in the sense that ∇' and ∇ agree on S).*

Proof. Let $\mathcal{D}_x = \text{kernel } \mathcal{A}_x$ for each x of the open submanifold U of M . If $X_x \in \mathcal{D}_x$ and $Y_x \in M_x$, then from

$$(20) \quad g(\mathcal{A}(X_x), Y_x) = g(X_x, \mathcal{A}(Y_x)) = 0$$

we see that $\mathcal{A}(M_x) = \mathcal{D}_x^\perp$ (in the horizontal subbundle). Suppose that X^1, \dots, X^n are C^∞ vector fields which form a basis of M_p at each p in a neighborhood of x . Then from the set

$$(21) \quad \{\mathcal{A}(X^1), \dots, \mathcal{A}(X^n)\}$$

we may select a minimal subset which spans \mathcal{D}_x^\perp . Since these are C^∞ and linearly independent at x , they are linearly independent and thus span \mathcal{D}_p^\perp at each p in some neighborhood of x . Consequently \mathcal{D}^\perp is a C^∞ distribution of dimension l on U , and hence $(\mathcal{D}^\perp)^\perp = \mathcal{D}$ is a C^∞ $(n-l)$ -dimensional C^∞ distribution on U . Suppose that $X, Y \in \mathcal{D}$. Then

$$(22) \quad \begin{aligned} (\nabla_X \mathcal{A})(Y) &= \nabla_X \mathcal{A}(Y) - \mathcal{A}(\nabla_X Y), \\ (\nabla_Y \mathcal{A})(X) &= \nabla_Y \mathcal{A}(X) - \mathcal{A}(\nabla_Y X). \end{aligned}$$

Since the vertical component of R' vanishes, the Codazzi equation $(\nabla_X \mathcal{A})(Y) = (\nabla_Y \mathcal{A})(X)$ holds, and since $\mathcal{A}(X) = \mathcal{A}(Y) = 0$ we have

$$(23) \quad \mathcal{A}(\nabla_X Y - \nabla_Y X) = 0.$$

However, since the AS -structure is Riemannian, $\nabla_X Y - \nabla_Y X = [X, Y]$, and thus

$$(24) \quad \mathcal{A}([X, Y]) = 0.$$

Thus \mathcal{D} is an involutive distribution on U , and consequently through each $x \in U$ there passes a maximal integral $(n-l)$ -dimensional manifold S of \mathcal{D} . Since \mathcal{A} vanishes on tangent vectors to S and

$$(25) \quad g(\mathcal{A}(X), Y) = h(X, Y), \quad \alpha(X, Y) = h(X, Y)\xi,$$

we see that the second fundamental form α of the AS -structure vanishes on tangent vectors to S , and hence that each geodesic of S is also a second order geodesic of S .

Remark. If we define in the usual manner

$$(26) \quad R'(W, X, Y, Z) = g(R'(Y, Z)X, W)$$

for $W, X, Y, Z \in \mathfrak{X}(M)$, and the AS -structure is Riemannian, we see that if

$$(27) \quad \frac{R'(X, Y, X, Y)}{g(X, X)g(Y, Y) - g^2(X, Y)} = k$$

for all $X, Y \in \mathfrak{X}(M)$, then

$$(28) \quad R'(X, Y)Z = k(g(Z, Y)X - g(Z, X)Y),$$

and consequently the vertical component of $R'(X, Y)Z$ vanishes. Hence the condition on the vertical component of $R'(X, Y)Z$ in Theorem 5 may be replaced with the above "constant curvature" condition (27).

Theorem 6. *Suppose that M bears a Riemannian AS-structure of pseudocodimension 1, with the properties that \mathcal{A} is parallel with respect to the induced first order connection on M , and that the type number $t(x)$ of \mathcal{A} at each point of an open neighborhood U of M is constantly 1. Then the conclusion of Theorem 5 holds.*

Proof. Suppose that $S, Y \in \mathcal{D}$ where \mathcal{D} is defined as in Theorem 5. If \mathcal{A} is parallel with respect to the induced first order connection on M , then

$$(29) \quad \nabla_X \mathcal{A}(Y) = \mathcal{A}(\nabla_X Y), \quad \nabla_Y \mathcal{A}(X) = \mathcal{A}(\nabla_Y X).$$

Since $X, Y \in \mathcal{D}$, it follows that $\mathcal{A}(\nabla_X Y) = \mathcal{A}(\nabla_Y X) = 0$, and hence that $\mathcal{A}(\nabla_X Y - \nabla_Y X) = \mathcal{A}([X, Y]) = 0$ due to the fact that the AS-structure is Riemannian. Thus \mathcal{D} is an involutive C^∞ distribution on U (that \mathcal{D} is C^∞ follows exactly as in Theorem 5) and the conclusion desired follows as in Theorem 5.

5. The mean curvature vector

The mean curvature vector of a manifold bearing an AS-structure is giving by

$$(30) \quad \eta = \text{tr } \mathcal{A}_i \xi_i,$$

where $\text{tr } \mathcal{A}_i$ denotes the trace of the map $X \rightarrow \mathcal{A}_i(X)$, ξ_i is an orthonormal basis of the vertical subbundle of 2M (which exists locally at least), and \mathcal{A}_i is defined by

$$(31) \quad g(\mathcal{A}_i(X), Y) = g(\alpha(X, Y), \xi_i).$$

We first note that in the case of an AS-structure of pseudocodimension 1, α and η are linearly dependent.

Theorem 7. *If a manifold M bears an AS-structure of pseudocodimension 1, then there exists a C^∞ map $h: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow R$ such that*

$$\alpha(X, Y) = h(X, Y)\eta / \|\eta\|$$

at each point of M except where $\eta = 0$ and $\alpha \neq 0$.

Proof. Suppose that ξ_1, \dots, ξ_n form an orthonormal basis of the vertical subbundle such that η and ξ_1 are linearly dependent. Since $g(\mathcal{A}_i(X), Y) = g(\alpha(X, Y), \xi_i)$, we see that $\mathcal{A}_i = 0$ and consequently $\text{tr } \mathcal{A}_i = 0, i = 2, \dots, n$. Thus

$$(32) \quad \eta = \text{tr } \mathcal{A}_i \xi_i = \text{tr } \mathcal{A}_1 \xi_1.$$

On the other hand

$$\alpha(X, Y) = g(\alpha(X, Y), \xi_i)\xi_i = g(\mathcal{A}_i(X), Y)\xi_i = g(\mathcal{A}_1(X), Y)\xi_1,$$

and $\xi_1 = \eta/\text{tr } \mathcal{A}_1 = n/\|\eta\|$ for $\eta \neq 0$. Thus taking $h(X, Y) = g(\mathcal{A}_1(X), Y)$ we have

$$\alpha(X, Y) = h(X, Y)\eta/\|\eta\|$$

at each point of M where $\eta \neq 0$. If $\alpha = 0$, then $\mathcal{A}_i = 0$, $i = 1, \dots, h$ and $\eta = 0$; hence $h(X, Y) = 0$. If we define $\eta/\|\eta\| = 0$ when $\eta = 0$, we again obtain the desired formula.

Suppose that we define the mean curvature of a manifold bearing an AS -structure by

$$(33) \quad H(x) = \|\eta_x\|,$$

and define \mathcal{A} by

$$(34) \quad g(\mathcal{A}(X), Y) = g(\alpha(X, Y), \eta/\|\eta\|)$$

for $\eta \neq 0$ and $\mathcal{A} = 0$ for $\eta = 0$. Then \mathcal{A} is C^∞ except where $\eta = 0$ and $\alpha \neq 0$, as in Theorem 7 we have

$$g(\mathcal{A}(X), Y) = g(h(X, Y)\eta/\|\eta\|, \eta/\|\eta\|) = h(X, Y)$$

except when $\eta = 0$ and $\alpha \neq 0$.

Theorem 8. *If M is a manifold bearing an AS -structure, then*

$$H(x) = \text{tr } \mathcal{A}_x.$$

Proof. If x is a point of M such that $\eta_x \neq 0$, then

$$\begin{aligned} g(\mathcal{A}(X_x), Y_x) &= g(\alpha(X_x, Y_x), (\text{tr } \mathcal{A}_i)\xi_i/\|\eta_x\|) \\ &= (\text{tr } \mathcal{A}_i)g(\alpha(X_x, Y_x), \xi_i)/\|\eta_x\| \\ &= (\text{tr } \mathcal{A}_i)g(\mathcal{A}_i(X_x), Y_x)/\|\eta_x\|. \end{aligned}$$

Thus

$$(35) \quad \mathcal{A} = (\text{tr } \mathcal{A}_i)\mathcal{A}_i/\|\eta\|,$$

and hence that

$$(36) \quad \text{tr } \mathcal{A} = (1/\|\eta_x\|) \sum_{i=1}^n (\text{tr } \mathcal{A}_i)^2 = \|\eta_x\| = H(x).$$

If $\eta_x = 0$, then from (34) we see that $\mathcal{A}_x = 0$ and hence that once again $H(x) = \text{tr } \mathcal{A}_x$.

In the case where there are no points of M such that $\eta = 0$ and $\alpha \neq 0$, we may endow M with an AS -structure of pseudocodimension 1 via the formulas

$$\begin{aligned} \nabla'_X Y &= \nabla_X Y + g(\alpha(X, Y), \eta / \|\eta\|) \eta / \|\eta\|, \\ \nabla'_X \eta / \|\eta\| &= -\mathcal{A}(X) + D_X \eta / \|\eta\|, \end{aligned}$$

and the mean curvature is the same as that of the original AS -structure.

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